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# BOUNDEDNESS OF THE WEYL FRACTIONAL INTEGRAL ON ONE-SIDED WEIGHTED LEBESGUE AND LIPSCHITZ SPACES

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## Abstract

In this paper we introduce the one-sided weighted spaces  $\mathcal{L}_w^-(\beta)$ ,  $-1 < \beta < 1$ . The purpose of this definition is to obtain an extension of the Weyl fractional integral operator  $I_\alpha^+$  from  $L_w^p$  into a suitable weighted space.

Under certain condition on the weight  $w$ , we have that  $\mathcal{L}_w^-(0)$  coincides with the dual of the Hardy space  $H_-^1(w)$ . We prove for  $0 < \beta < 1$ , that  $\mathcal{L}_w^-(\beta)$  consists of all functions satisfying a weighted Lipschitz condition. In order to give another characterization of  $\mathcal{L}_w^-(\beta)$ ,  $0 \leq \beta < 1$ , we also prove a one-sided version of John-Nirenberg Inequality.

Finally, we obtain necessary and sufficient conditions on the weight  $w$  for the boundedness of an extension of  $I_\alpha^+$  from  $L_w^p$  into  $\mathcal{L}_w^-(\beta)$ ,  $-1 < \beta < 1$ , and its extension to a bounded operator from  $\mathcal{L}_w^-(0)$  into  $\mathcal{L}_w^-(\alpha)$ .

## 1. Notations, definitions and prerequisites

Let  $E \subseteq \mathbb{R}$  be a Lebesgue measurable set. We shall denote its Lebesgue measure by  $|E|$  and the characteristic function of  $E$  by  $\chi_E$ .

As usual, a weight  $w$  is a measurable, non-negative and locally integrable function defined on  $\mathbb{R}$ .

Let  $w$  be a weight. Given a Lebesgue measurable set  $E \subseteq \mathbb{R}$ , its  $w$ -measure will be denote by  $w(E) = \int_E w(t) dt$ .

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Let  $1 < p < \infty$ . The weight  $w$  belongs to the class  $A_p^-$  if there exists a constant  $C$  such that

$$\sup_{h>0} \left[ \frac{1}{h^p} \int_a^{a+h} w(x) dx \left( \int_{a-h}^a w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \right] \leq C,$$

for all real number  $a$ . In a similar way,  $w$  belongs to  $A_p^+$  if

$$\sup_{h>0} \left[ \frac{1}{h^p} \int_{a-h}^a w(x) dx \left( \int_a^{a+h} w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \right] \leq C,$$

for all real number  $a$ . The class  $A_1^-$  is defined by the condition

$$\sup_{h>0} \left[ \frac{1}{h} \int_a^{a+h} w(x) dx \right] \leq Cw(a),$$

for almost every real number  $a$ . The weight  $w$  belongs to  $A_1^+$  if

$$\sup_{h>0} \left[ \frac{1}{h} \int_{a-h}^a w(x) dx \right] \leq Cw(a),$$

for almost every  $a$ . These classes  $A_p^-$  and  $A_p^+$  were introduced by E. Sawyer in [12]. We recall three basic results on these weights.

- (i) For  $1 < p < \infty$ , a weight  $w$  belongs to  $A_p^-$  if and only if  $w^{1-p'}$  belongs to  $A_{p'}^+$ , where  $\frac{1}{p} + \frac{1}{p'} = 1$ .
- (ii) If  $1 \leq p < q < \infty$ , then  $A_p^- \subset A_q^-$ .
- (iii) If  $1 < p < \infty$  and  $w$  belongs to  $A_p^-$ , then  $w$  belongs to  $A_{p-\epsilon}^-$  for some  $\epsilon > 0$ .

The proof of (i) and (ii) are very simple and (iii) can be found in Proposition 3 in [3].

In the sequel, for each bounded interval  $I = [a, b]$  we shall denote  $I^- = [a - |I|, a]$  and  $I^+ = [b, b + |I|]$ .

Let  $1 \leq q < \infty$ . A weight  $w$  satisfies the condition  $RH^-(q)$  if there exists a constant  $C$  such that for every bounded interval  $I$ .

$$\left[ \frac{1}{|I|} \int_I w(x)^q dx \right]^{1/q} \leq C \frac{1}{|I|} \int_{I^-} w(x) dx.$$

We shall say that a weight  $w$  belongs to  $D^-$  if there exists a constant  $C$  such that for every bounded interval  $I$ ,

$$w(I \cup I^+) \leq Cw(I).$$

It is well known that if  $w \in A_p^-$ ,  $1 \leq p < \infty$ , then  $w \in D^-$ .

Let  $w$  be a weight,  $1 \leq p < \infty$  and  $f$  a measurable function. We shall say that  $f$  belongs to  $L_w^p$  if

$$\|f\|_{p,w}^p = \int_{-\infty}^{\infty} \left[ \frac{|f(x)|}{w(x)} \right]^p dx$$

is finite. The function  $f$  belongs to  $\widetilde{L}_w^p$  if

$$[f]_{p,w}^p = \sup_{t>0} t^p \left| \left\{ x \in \mathbb{R} : \frac{|f(x)|}{w(x)} > t \right\} \right|$$

is finite.

Let  $0 < \alpha < 1$ . Given  $f$  a measurable function on  $\mathbb{R}$ , its Weyl fractional integral is defined by

$$I_\alpha^+ f(x) = \int_x^\infty \frac{f(y)}{(y-x)^{1-\alpha}} dy,$$

whenever this integral is finite.

In the sequel, the letter  $C$  will denote a positive finite constant not necessarily the same at each occurrence. If  $1 \leq p \leq \infty$  then  $p'$  will be its conjugate exponent, that is,  $1/p + 1/p' = 1$ .

Let  $w$  be a weight and  $-1 < \beta < 1$ .

**Definition 1.1.** We say that a locally integrable function  $f$  defined on  $\mathbb{R}$  belongs to  $\mathcal{L}_w(\beta)$ , if there exists a constant  $C$  such that

$$\frac{1}{w(I)|I|^\beta} \int_I |f(y) - f_I| dy \leq C,$$

for every bounded interval  $I$ , where  $f_I = \frac{1}{|I|} \int_I f$ . The least constant  $C$  will be denoted  $\|f\|_{\mathcal{L}_w(\beta)}$ .

The spaces  $\mathcal{L}_w(\beta)$  were introduced by E. Harboure, O. Salinas and B. Viviani in [1]. They are a weighted version of the spaces  $\mathcal{L}_{\lambda,p}$ , for  $p = 1$ , defined by J. Peetre in [8]. If  $w$  belongs to  $A_q^-$ ,  $1 \leq q < 2$ , then  $\mathcal{L}_w(0)$  is the dual space of the one-sided weighted Hardy space  $H_-^1(w)$ , see [10] and [11].

**Definition 1.2.** We say that a locally integrable function  $f$  defined on  $\mathbb{R}$  belongs to  $\mathcal{L}_w^-(\beta)$ , if there exists a constant  $C$  such that

$$\frac{1}{w(I^-)|I|^\beta} \int_I |f(y) - f_I| dy \leq C,$$

for every bounded interval  $I$ . The least constant  $C$  satisfying this inequality will be denoted  $\|f\|_{\mathcal{L}_w^-(\beta)}$ .

In the following definition, we consider a one-sided version of the classes  $H(\alpha, p)$  defined in [1].

**Definition 1.3.** Let  $0 < \alpha < 1$  and  $1 < p \leq \infty$ . We say that a weight  $w$  belongs to  $H^-(\alpha, p)$  if there exists a constant  $C$  such that for every bounded interval  $I = [a, b]$ , the inequality

$$|I|^{\frac{1}{p} - \alpha + 1} \left[ \int_b^\infty \frac{w(y)^{p'}}{(y - a)^{(2-\alpha)p'}} dy \right]^{1/p'} \leq C \frac{w(I)}{|I|},$$

holds.

## 2. Statement of the main results

Lemma 4.1(iii) shows that if  $w$  belongs to  $H^-(\alpha, p)$ ,  $1 < p \leq \infty$ , then  $w$  belongs to  $D^-$  and therefore  $\mathcal{L}_w(\beta) \subseteq \mathcal{L}_w^-(\beta)$  for every  $\beta$ :  $-1 < \beta < 1$ . The next theorem states that  $w$  belonging to  $D^-$  is a sufficient condition for the equality of these spaces, whenever  $0 \leq \beta < 1$ .

**Theorem 2.1.** *Let  $0 \leq \beta < 1$  and let  $w$  belong to  $D^-$ . Then, the spaces  $\mathcal{L}_w(\beta)$  and  $\mathcal{L}_w^-(\beta)$  are equal, and their norms are equivalent.*

The next theorem gives us a characterization of the spaces  $\mathcal{L}_w(\beta)$ ,  $0 \leq \beta < 1$ , whenever  $w$  belongs to  $A_p^-$ . In the case  $\beta = 0$ , we shall prove this result using Proposition 3.6, which states a one-sided weighted version of John-Nirenberg Inequality.

**Theorem 2.2.** *Let  $0 \leq \beta < 1$  and  $1 \leq p < \infty$ . Let  $w$  be a weight such that  $w$  belongs to  $A_p^-$ . Then,  $f \in \mathcal{L}_w(\beta)$  if and only if there exists a constant  $C$  such that*

$$(2.1) \quad \int_{I^-} |f(x) - f_{I^+}|^q w(x)^{1-q} dx \leq C w(I^-) |I|^{\beta q},$$

for all bounded interval  $I$  and every  $q$ :  $1 \leq q \leq p'$ ,  $q < \infty$ .

The following two theorems state a sufficient and necessary condition on the weight  $w$  to obtain extensions of  $I_\alpha^+$  defined on certain spaces.

**Theorem 2.3.** *Let  $0 < \alpha < 1$ ,  $1 < p < \infty$  and  $\beta = \alpha - 1/p$ . The following statements are equivalent.*

- (i) *The weight  $w$  belongs to  $H^-(\alpha, p)$ .*
- (ii) *The operator  $I_\alpha^+$  can be extended to a linear bounded operator  $\widetilde{I}_\alpha^+$  from  $\widetilde{L}_w^p$  into  $\mathcal{L}_w^-(\beta)$  by means of*

$$(2.2) \quad \begin{aligned} \widetilde{I}_\alpha^+(f)(x) = & - \int_{x_0}^x \frac{f(y) dy}{|y-x|^{1-\alpha}} \\ & + \int_{x_0}^\infty \left[ \frac{1}{|y-x|^{1-\alpha}} - \frac{1 - \chi_{[x_0, x_0+1]}(y)}{(y-x_0)^{1-\alpha}} \right] f(y) dy, \end{aligned}$$

for any  $x_0 \in \mathbb{R}$ .

- (iii) *The operator  $I_\alpha^+$  can be extended to a linear bounded operator  $\widetilde{I}_\alpha^+$  from  $\widetilde{L}_w^p$  into  $\mathcal{L}_w^-(\beta)$ , where  $\widetilde{I}_\alpha^+$  is defined as in (2.2).*

**Theorem 2.4.** *Let  $w$  a weight and  $0 < \alpha < 1$ . The following statements are equivalent.*

- (i) *The weight  $w$  belongs to  $H^-(\alpha, \infty)$ .*
- (ii) *The operator  $I_\alpha^+$  can be extended to a linear bounded operator  $\widetilde{I}_\alpha^+ : \mathcal{L}_w(0) \rightarrow \mathcal{L}_w(\alpha)$  by means of*

$$\widetilde{I}_\alpha^+(f)(x) = \int_{-\infty}^\infty \left[ \frac{\chi_{[x_0, \infty)}(y)}{|y-x_0|^{1-\alpha}} - \frac{\chi_{[x, \infty)}(y)}{|y-x|^{1-\alpha}} \right] f(y) dy,$$

for an appropriate choice of  $x_0 \in \mathbb{R}$ .

**Remark 2.5.** Let  $1 < p < \frac{1}{\alpha}$  and  $\beta = \alpha - 1/p < 0$ .

- (i) It is easy to see that if  $w$  belongs to  $RH^-(\frac{1}{1+\beta})$ , then  $L_w^{-1/\beta} \subseteq \mathcal{L}_w^-(\beta)$ .
- (ii) By Lemma 4.4 in [9], if  $w^{p'}$  belongs to  $A_{-\beta p'+1}^-$  then  $w$  satisfies the condition  $RH^-(p')$ , and taking into account that  $\frac{1}{1+\beta} < p'$ , it follows that  $w$  belongs to  $RH^-(\frac{1}{1+\beta})$ .
- (iii) Theorem 6 in [4] states the fact that  $w^{p'}$  belongs to  $A_{-\beta p'+1}^-$  is a necessary and sufficient condition for the boundedness of  $I_\alpha^+$  from  $L_w^p$  into  $L_w^{-1/\beta} \subseteq \mathcal{L}_w^-(\beta)$ .
- (iv) If  $w^{p'}$  belongs to  $A_{-\beta p'+1}^-$ , since  $w^{p'} \in A_{p'+1}^-$ , we have that  $w$  belongs to  $H^-(\alpha, p)$ . However, there exist weights  $w$  belonging to  $H^-(\alpha, p)$  such that  $w^{p'}$  does not belong to  $A_{p'+1}^-$ , for example,  $w(x) = |x|^\gamma$  for  $-\beta \leq \gamma < 1 - \beta$ , see Remark 4.3.

In consequence, if  $-1 < \beta < 0$  and  $w^{p'}$  belongs to  $A_{-\beta p'+1}^-$ , the extension of  $I_\alpha^+$  in Theorem 2.3 can be obtained from Theorem 6 in [4]. But, (iv) shows that Theorem 2.3 can be applied to a larger class of weights.

*Remark 2.6.* Let  $w$  be a weight. We shall say that a locally integrable function  $f$  defined on  $\mathbb{R}$ , belongs to  $MW^-(w)$  if there exists a constant  $C$  such that

$$\frac{1}{|I|} \frac{1}{\operatorname{ess\,inf}_{I^-} w} \int_I |f(y) - f_I| dy \leq C,$$

for every bounded interval  $I$ .

- (i) By Definition 1.2, it follows that  $MW^-(w) \subseteq \mathcal{L}_w^-(0)$ . Moreover, if  $w$  belongs to  $A_1^-$  then  $\mathcal{L}_w(0) \subseteq MW^-(w)$ , and as a consequence of Theorem 2.1,  $\mathcal{L}_w^-(0) = MW^-(w)$ .
- (ii) Following the same lines of Theorem 7 in [7], it can be seen that, in the case  $\alpha = 1/p$ , the weight  $w^{p'}$  belongs to  $A_1^-$  if and only if the operator  $I_\alpha^+$  is bounded from  $L_w^p$  into  $MW^-(w)$ . Also see [2].
- (iii) If  $w^{p'}$  belongs to  $A_1^-$  then, by Remark 4.3,  $w$  belongs to  $H^-(\alpha, p)$ .

In consequence, the fact that  $w^{p'}$  belongs to  $A_1^-$  implies the boundedness of  $I_\alpha^+$  from  $L_w^p$  into  $MW^-(w)$ , is contained in Theorem 2.3.

### 3. The spaces $\mathcal{L}_w(\beta)$ and $\mathcal{L}_w^-(\beta)$

The next lemma will be used in the proof of Theorem 2.1.

**Lemma 3.1.** *Let  $-1 < \beta < 1$ ,  $f$  a locally integrable function defined on  $\mathbb{R}$ , and  $w \in D^-$ . The following statements are equivalent.*

- (i)  $f \in \mathcal{L}_w^-(\beta)$ .
- (ii) *There exists a constant  $C$  such that for every  $a \in \mathbb{R}$  and  $h > 0$ ,*

$$\frac{1}{w([a - h/2, a])h^\beta} \int_a^{a+h} |f(y) - f_{[a+h/2, a+h]}| dy \leq C.$$

- (iii) *There exists a constant  $C$  such that for every  $a \in \mathbb{R}$  and  $h > 0$ ,*

$$\frac{1}{w([a - h/2, a])h^\beta} \int_a^{a+h} |f(y) - f_{[a+h, a+3h]}| dy \leq C.$$

*The constants  $C$  in (ii) and (iii) are equivalent to  $\|f\|_{\mathcal{L}_w^-(\beta)}$ .*

*Proof:* (i)  $\Rightarrow$  (ii). Using (i) and taking into account that  $w \in D^-$ , we have

$$\begin{aligned}
& \int_a^{a+h/2} |f(y) - f_{[a+h/2, a+h]}| dy \\
& \leq \int_a^{a+h/2} |f(y) - f_{[a+h/4, a+h/2]}| dy + 2 \int_{a+h/4}^{a+h} |f(y) - f_{[a+h/2, a+h]}| dy \\
& \leq 3 \int_a^{a+h/2} |f(y) - f_{[a, a+h/2]}| dy + 5 \int_{a+h/4}^{a+h} |f(y) - f_{[a+h/4, a+h]}| dy \\
& \leq C \|f\|_{\mathcal{L}_w^-(\beta)} w([a - h/2, a]) h^\beta + C \|f\|_{\mathcal{L}_w^-(\beta)} w([a - h/2, a + h/4]) h^\beta \\
& \leq C \|f\|_{\mathcal{L}_w^-(\beta)} w([a - h/2, a]) h^\beta.
\end{aligned}$$

From these inequalities and using (i) again, we have the estimate

$$\begin{aligned}
& \int_a^{a+h} |f(y) - f_{[a+h/2, a+h]}| dy \\
& = \int_a^{a+h/2} |f(y) - f_{[a+h/2, a+h]}| dy + \int_{a+h/2}^{a+h} |f(y) - f_{[a+h/2, a+h]}| dy \\
& \leq C \|f\|_{\mathcal{L}_w^-(\beta)} w([a - h/2, a]) h^\beta + C \|f\|_{\mathcal{L}_w^-(\beta)} w([a, a + h/2]) h^\beta \\
& \leq C \|f\|_{\mathcal{L}_w^-(\beta)} w([a - h/2, a]) h^\beta,
\end{aligned}$$

which shows that (ii) holds. In a similar way it can be proved that (ii)  $\Rightarrow$  (iii) and (iii)  $\Rightarrow$  (i).  $\square$

As we have already mentioned if  $w$  belongs to  $D^-$  then, for every  $-1 < \beta < 1$  we have the inclusion  $\mathcal{L}_w(\beta) \subseteq \mathcal{L}_w^-(\beta)$ . In order to prove Theorem 2.1, it will be sufficient to show that  $\mathcal{L}_w^-(\beta) \subseteq \mathcal{L}_w(\beta)$ .

*Proof of Theorem 2.1:* We suppose that  $f \in \mathcal{L}_w^-(\beta)$ . Let  $a \in \mathbb{R}$  and  $h > 0$ . For each  $j \geq 0$  we define  $a_j = a + h/2^j$ . Then,

$$\begin{aligned}
 & \int_a^{a+h/2} |f(y) - f_{[a+h/2, a+h]}| dy \\
 (3.1) \quad &= \sum_{j=1}^{\infty} \int_{a_{j+1}}^{a_j} |f(y) - f_{[a+h/2, a+h]}| dy \\
 &\leq \sum_{j=1}^{\infty} \int_{a_{j+1}}^{a_j} |f(y) - f_{[a_j, a_{j-1}]}| dy + \sum_{j=2}^{\infty} \frac{h}{2^{j+1}} |f_{[a_j, a_{j-1}]} - f_{[a_1, a_0]}| \\
 &= I + II.
 \end{aligned}$$

Taking into account that for each  $j \geq 2$ ,

$$|f_{[a_j, a_{j-1}]} - f_{[a_1, a_0]}| \leq \frac{2^j}{h} \int_{a_j}^{a_{j-1}} |f - f_{[a+h/2, a+h]}|$$

it follows that,

$$II \leq \sum_{j=2}^{\infty} \frac{1}{2} \int_{a_j}^{a_{j-1}} |f - f_{[a+h/2, a+h]}| = \frac{1}{2} \int_a^{a+h/2} |f(y) - f_{[a+h/2, a+h]}| dy.$$

Then, by (3.1)

$$(3.2) \quad \int_a^{a+h/2} |f(y) - f_{[a+h/2, a+h]}| dy \leq 2I.$$

Now, using (iii) of Lemma 3.1 and keeping in mind that  $\beta \geq 0$  we have that,

$$(3.3) \quad I \leq C \sum_{j=1}^{\infty} \left( \frac{h}{2^j} \right)^{\beta} w([a_{j+2}, a_{j+1}]) \leq Ch^{\beta} w([a, a + h/4]).$$



From (3.2) and (3.3), and taking into account that  $f \in \mathcal{L}_w^-(\beta)$ , we get

$$\begin{aligned}
& \int_a^{a+h} |f(y) - f_{[a+h/2, a+h]}| dy \\
&= \int_a^{a+h/2} |f(y) - f_{[a+h/2, a+h]}| dy + \int_{a+h/2}^{a+h} |f(y) - f_{[a+h/2, a+h]}| dy \\
&\leq Ch^\beta w([a, a+h/4]) + Ch^\beta w([a, a+h/2]) \\
&\leq Ch^\beta w([a, a+h]).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \int_a^{a+h} |f(y) - f_{[a, a+h]}| dy \\
&\leq 3 \int_a^{a+h} |f(y) - f_{[a+h/2, a+h]}| dy \leq Ch^\beta w([a, a+h]),
\end{aligned}$$

which shows that  $f \in \mathcal{L}_w(\beta)$ .  $\square$

*Remark 3.2.* Let  $-1 < \beta < 0$  and  $w(t) = e^{-t}$ . The weight  $w$  belongs to  $A_1^-$  however, we only have the strict inclusion  $\mathcal{L}_w(\beta) \subset \mathcal{L}_w^-(\beta)$ . For example, given  $a > 1$  we consider the function

$$f(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 1, & t < 0. \end{cases}$$

We observe, using Remark 2.5(i), that  $f \in \mathcal{L}_w^-(\beta)$ . On the other hand,

$$\begin{aligned}
\frac{1}{h^\beta w([0, h])} \int_0^h |f - f_{[h, 2h]}| &= \frac{1}{h^\beta (1 - e^{-h})} \left[ \frac{1 - e^{-ah}}{a} - \frac{e^{-ah}}{a} (1 - e^{-ah}) \right] \\
&= \frac{(1 - e^{-ah})^2}{h^\beta (1 - e^{-h}) a},
\end{aligned}$$

which tends to infinite whenever  $h$  tends to infinite. This implies that  $f \notin \mathcal{L}_w(\beta)$ .

The next proposition will be used in the proof of Theorem 2.2.

**Proposition 3.3.** *Let  $0 < \beta < 1$  and let  $w$  belong to  $D^-$ . Then,  $f \in \mathcal{L}_w(\beta)$  if and only if, there exists a constant  $C$  such that*

$$(3.4) \quad |f(x) - f(y)| \leq C \left[ \int_x^{x + \frac{|y-x|}{2}} \frac{w(z)}{(z-x)^{1-\beta}} dz + \int_y^{y + \frac{|y-x|}{2}} \frac{w(z)}{(z-y)^{1-\beta}} dz \right],$$

for almost every real numbers  $x$  and  $y$ .

*Proof:* We suppose that  $f \in \mathcal{L}_w(\beta)$ . We shall show that for every  $h > 0$  and almost every  $x$ ,

$$(3.5) \quad |f(x) - f_{[x+h/2, x+h]}| \leq C \int_x^{x+h/2} \frac{w(z)}{(z-x)^{1-\beta}} dz.$$

For each  $i \geq 0$  let  $x_i = x + h/2^i$ . If  $x$  is a Lebesgue point of  $f$  we have that,

$$(3.6) \quad \begin{aligned} |f(x) - f_{[x+h/2, x+h]}| &\leq |f(x) - f_{[x_{i+1}, x_i]}| + |f_{[x_{i+1}, x_i]} - f_{[x_1, x_0]}| \\ &\leq |f(x) - f_{[x_{i+1}, x_i]}| + \sum_{j=1}^i |f_{[x_{j+1}, x_j]} - f_{[x_j, x_{j-1}]}| \\ &\leq \sum_{j=1}^{\infty} |f_{[x_{j+1}, x_j]} - f_{[x_j, x_{j-1}]}|. \end{aligned}$$

For each  $j \geq 1$ , since  $f \in \mathcal{L}_w(\beta)$  we obtain

$$|f_{[x_{j+1}, x_j]} - f_{[x_j, x_{j-1}]}| \leq C \frac{1}{(x_{j+1} - x_{j-1})^{1-\beta}} w([x_{j+1}, x_{j-1}]).$$

From this inequality, (3.6) and taking into account that  $w \in D^-$  we get,

$$\begin{aligned} |f(x) - f_{[x+h/2, x+h]}| &\leq C \sum_{j=1}^{\infty} \int_{x_{j+1}}^{x_{j-1}} \frac{w(z)}{(z-x)^{1-\beta}} dz \\ &= C \int_x^{x+h} \frac{w(z)}{(z-x)^{1-\beta}} dz \\ &\leq C \int_x^{x+h/2} \frac{w(z)}{(z-x)^{1-\beta}} dz, \end{aligned}$$

which shows that (3.5) holds. Let  $x < y$  two Lebesgue points of  $f$ . By (3.5) we have that,

$$\begin{aligned}
 |f(x) - f(y)| &\leq |f(x) - f_{[\frac{x+y}{2}, y]}| + |f(y) - f_{[y + \frac{y-x}{2}, y + (y-x)]}| \\
 &\quad + |f_{[\frac{x+y}{2}, y]} - f_{[y + \frac{y-x}{2}, y + (y-x)]}| \\
 (3.7) \quad &\leq C \left[ \int_x^{x + \frac{|y-x|}{2}} \frac{w(z)}{(z-x)^{1-\beta}} dz + \int_y^{y + \frac{|y-x|}{2}} \frac{w(z)}{(z-y)^{1-\beta}} dz \right] \\
 &\quad + |f_{[\frac{x+y}{2}, y]} - f_{[y + \frac{y-x}{2}, y + (y-x)]}|.
 \end{aligned}$$

From the hypotheses  $f \in \mathcal{L}_w(\beta)$  and  $w \in D^-$ , it follows that the third term on the right hand is bounded by

$$\begin{aligned}
 &\frac{C}{y-x} \int_{x + \frac{y-x}{2}}^{y+(y-x)} |f(t) - f_{[x + \frac{y-x}{2}, y + (y-x)]}| dt \\
 &\leq \frac{C}{(y-x)^{1-\beta}} w([x, y + \frac{y-x}{2}]) \leq C \int_x^{(x+y)/2} \frac{w(z)}{(z-x)^{1-\beta}} dz.
 \end{aligned}$$

Therefore, by (3.7) we have that (3.4) holds.

Conversely, given a real number  $a$  and  $h > 0$ , by (3.4)

$$\begin{aligned}
 (3.8) \quad &\int_a^{a+h} |f(x) - f_{[a, a+h]}| dx \\
 &\leq C \left[ \int_a^{a+h} \int_x^{x + \frac{|y-x|}{2}} \frac{w(z)}{(z-x)^{1-\beta}} dz dx + \int_a^{a+h} \int_y^{y + \frac{|y-x|}{2}} \frac{w(z)}{(z-y)^{1-\beta}} dz dy \right].
 \end{aligned}$$

Changing the order of integration and taking into account that  $w \in D^-$ , it follows that (3.8) is bounded by  $Ch^\beta w([a, a+h])$ . This completes the proof of the proposition.  $\square$

The next two lemmas will be needed in the proof of Proposition 3.6.

**Lemma 3.4.** *Let  $w \in D^-$  and  $f \in \mathcal{L}_w(0)$ . Given two intervals  $I \subseteq J$  the inequality*

$$\frac{1}{w(J)} \int_J |f(y) - f_I + \chi_{I \cup I}(y)| dy \leq C \|f\|_{\mathcal{L}_w(0)},$$

*holds with a constant  $C$  only depending on  $w$ .*

*Proof:* Let  $I = (a, b)$  and  $J = (c, d)$ . We consider  $\alpha = \max\{a - |I|, c\}$  and  $\beta = b + |I|$ . Since  $J \cap (I^- \cup I) \subseteq (\alpha, \beta)$  we have that,

$$(3.9) \quad \begin{aligned} & \frac{1}{w(J)} \int_J |f(y) - f_{I^+}| \chi_{I^- \cup I}(y) dy \leq \frac{1}{w(J)} \int_\alpha^\beta |f(y) - f_{I^+}| dy \\ & \leq \frac{1}{w(J)} \left[ \int_\alpha^\beta |f(y) - f_{(\alpha, \beta)}| dy + \frac{(\beta - \alpha)}{|I^+|} \int_{I^+} |f(y) - f_{(\alpha, \beta)}| dy \right]. \end{aligned}$$

We observe that  $(\beta - \alpha) \leq 3|I|$ , which implies

$$(3.9) \leq \frac{4}{w(J)} \int_\alpha^\beta |f(y) - f_{(\alpha, \beta)}| dy.$$

From the hypotheses  $f \in \mathcal{L}_w(0)$  and  $w \in D^-$ , and taking into account that  $(\alpha, \beta) \subseteq J \cup J^+$ , (3.9) is bounded by

$$\frac{4}{w(J)} \|f\|_{\mathcal{L}_w(0)} w((\alpha, \beta)) \leq C \|f\|_{\mathcal{L}_w(0)},$$

as we wanted to prove.  $\square$

**Lemma 3.5.** *Let  $1 < p < \infty$  and  $w \in A_p^-$ . Then, there exists a constant  $C$  such that for every  $\beta > 0$  the inequality*

$$(3.10) \quad w(\{x \in I^- : w(x) < \beta\}) \leq C \left[ \beta \frac{|I^+|}{w(I^+)} \right]^{p'} w(I^+),$$

*holds.*

*Proof:* This lemma is a simple variant of Lemma 3.1 in [6].  $\square$

The following result is a one-sided weighted version of John-Nirenberg Inequality. For its proof we shall use the method employed in Theorem 3 in [6] and the techniques of Lemma 1 in [5].

**Proposition 3.6.** *Let  $f$  belong to  $\mathcal{L}_w(0)$ . Then,*

- (i) *If  $w \in A_1^-$  there exist positive constants  $C_1$  and  $C_2$  such that for every  $\lambda > 0$ ,*

$$w(\{x \in I^- : |f(x) - f_{I^+}| w(x)^{-1} > \lambda\}) \leq C_1 e^{-C_2 \lambda / \|f\|_{\mathcal{L}_w(0)}} w(I^-)$$

*holds for every bounded interval  $I$ .*

- (ii) *If  $w \in A_p^-$ ,  $1 < p < \infty$  there exists a positive constant  $C_3$  such that for every  $\lambda > 0$ ,*

$$w(\{x \in I^- : |f(x) - f_{I^+}| w(x)^{-1} > \lambda\}) \leq C_3 (1 + \lambda / \|f\|_{\mathcal{L}_w(0)})^{-p'} w(I^-)$$

*holds for every bounded interval  $I$ .*

*Proof:* Without loss of generality we can suppose that  $\|f\|_{\mathcal{L}_w(0)} = 1$ . For each  $\lambda > 0$  and each bounded interval  $I$ , let

$$A(\lambda, I) = w(\{x \in I^- : |f(x) - f_{I^+}|w(x)^{-1} > \lambda\}),$$

and

$$(3.11) \quad \mathcal{A}(\lambda) = \sup \frac{A(\lambda, I)}{w(I^-)},$$

where the supremum is taken over all  $f : \|f\|_{\mathcal{L}_w(0)} = 1$ , and all bounded interval  $I$ . Thus, for every  $\lambda > 0$ , we have that  $\mathcal{A}(\lambda) \leq 1$ .

By Lemma 3.4 there exists a constant  $\mu$  satisfying

$$(3.12) \quad \frac{1}{w(J)} \int_J |f(y) - f_{I^+}| \chi_{I^- \cup I}(y) dy \leq \mu,$$

for every bounded intervals  $I \subseteq J$  and every  $f : \|f\|_{\mathcal{L}_w(0)} = 1$ .

Fixed  $I = [a, b]$ , let  $s > \mu$  and

$$\Omega_s = \{x \in \mathbb{R} : M_w^-(|f - f_{I^+}| \chi_{I^- \cup I} w^{-1})(x) > s\},$$

where  $M_w^-$  is the left sided maximal function with respect to the measure  $w$  defined as

$$M_w^-(g)(x) = \sup_{h>0} \frac{\int_{x-h}^x |g(y)| w(y) dy}{w([x-h, x])}.$$

Since  $\Omega_s$  is an open set, we can write  $\Omega_s = \cup_{i \geq 1} J_i$ , where the  $J_i$ 's are its connected components.

We observe that if  $J_i \cap I^- \neq \emptyset$  then  $J_i \cap I^+ = \emptyset$ . In fact, suppose that  $J_i \cap I^- \neq \emptyset$  and let  $J_i = (\alpha, \beta)$ . If  $\beta \geq b$  a simple variant of Lemma 2.1 in [12], shows that

$$\mu < s \leq \frac{1}{w((\alpha, b))} \int_{\alpha}^b |f(y) - f_{I^+}| \chi_{I^- \cup I}(y) dy.$$

However, using (3.12) we have that

$$\frac{1}{w((\alpha, b))} \int_{\alpha}^b |f(y) - f_{I^+}| \chi_{I^- \cup I}(y) dy \leq \mu.$$

In consequence,  $\beta < b$  and  $J_i \cap I^+ = \emptyset$ .

Let  $\{J_i : J_i \cap I^- \neq \emptyset\} = \{H_i\}_{i \geq 1}$ , where  $H_i = (a_i, b_i)$ . For each  $i$ , since  $M_w^- (|f - f_{I^+}| \chi_{I^- \cup I} w^{-1})(b_i) \leq s$  we have that,

$$(3.13) \quad H_i \subseteq I^- \cup I \quad \text{and} \quad \frac{1}{w(H_i)} \int_{H_i} |f(y) - f_{I^+}| dy = s.$$

By Lebesgue's Differentiation Theorem with respect to  $w$  for almost every  $x \in I^- \setminus \cup_{i \geq 1} H_i$ ,

$$|f(x) - f_{I+}|w(x)^{-1} \leq s.$$

Using (3.13), (3.12) and keeping in mind that  $w \in D^-$ , we obtained that

$$(3.14) \quad \sum_{i \geq 1} w(H_i) = \frac{1}{s} \sum_{i \geq 1} \int_{H_i} |f(y) - f_{I+}| dy \\ \leq \frac{1}{s} \int_{I^- \cup I} |f(y) - f_{I+}| dy \leq \frac{1}{s} \mu w(I^- \cup I) \leq \frac{1}{s} \mu C_w w(I^-).$$

Fixed  $H_i = (a_i, b_i)$  we define the sequences  $(x_k)_{k \geq 1}$  and  $(y_k)_{k \geq 1}$  by  $b_i - x_k = 2(b_i - y_k) = (2/3)^k |H_i|$ , and the intervals  $H_{i,k} = (x_k, y_k)$ . Therefore,

$$(3.15) \quad H_i = \bigcup_{k \geq 1} H_{i,k}^-, \quad \frac{1}{w(H_{i,k}^+)} \int_{H_{i,k}^+} |f(y) - f_{I+}| dy \leq s,$$

and

$$|f(x) - f_{I+}|w(x)^{-1} \leq \lambda \quad \text{a.e.} \quad x \in I^- \setminus \bigcup_{k,i} H_{i,k}^-.$$

Then,

$$A(\lambda, I) \leq \sum_{i,k} w(\{x \in H_{i,k}^- : |f(x) - f_{I+}|w(x)^{-1} > \lambda\}).$$

If  $\mu < s \leq \lambda$  and  $0 < \gamma < \lambda$ , we have that

$$(3.16) \quad A(\lambda, I) \leq \sum_{i,k} w(\{x \in H_{i,k}^- : |f(x) - f_{H_{i,k}^+}|w(x)^{-1} > \lambda - \gamma\}) \\ + \sum_{i,k} w(\{x \in H_{i,k}^- : |f_{H_{i,k}^+} - f_{I+}|w(x)^{-1} > \gamma\}) = I + II.$$

From (3.11), (3.15) and (3.14) we obtain the estimate

$$(3.17) \quad I \leq \sum_{i,k} \mathcal{A}(\lambda - \gamma) w(H_{i,k}^-) = \mathcal{A}(\lambda - \gamma) \sum_i w(H_i) \\ \leq \frac{C_w \mu}{s} \mathcal{A}(\lambda - \gamma) w(I^-).$$

On the other hand, (3.15) implies that

$$(3.18) \quad |f_{H_{i,k}^+} - f_{I+}| \leq \frac{1}{|H_{i,k}^+|} \int_{H_{i,k}^+} |f(y) - f_{I+}| dy \leq s \frac{w(H_{i,k}^+)}{|H_{i,k}^+|}.$$

If  $w \in A_1^-$  there exists  $\rho > 1$  such that for every  $i, k$  and almost every  $x \in H_{i,k}^-$ ,

$$\frac{w(H_{i,k}^+)}{|H_{i,k}^+|} \leq \rho w(x).$$

Then, using (3.18) we have

$$|f_{H_{i,k}^+} - f_{I^+}| \leq \rho s \operatorname{ess\,inf}_{x \in H_{i,k}^-} w(x).$$

In consequence,

$$\begin{aligned} & w(\{x \in H_{i,k}^- : |f_{H_{i,k}^+} - f_{I^+}| w(x)^{-1} > \gamma\}) \\ & \leq w\left(\left\{x \in H_{i,k}^- : w(x) < \frac{\rho s}{\gamma} \operatorname{ess\,inf}_{x \in H_{i,k}^-} w(x)\right\}\right). \end{aligned}$$

Choosing  $s = 2\mu C_w$  and  $\gamma = \rho s$ , if  $\lambda > \gamma$  we have  $\mu < s < \lambda$  and  $II = 0$ . Then, from (3.16) and (3.17) we obtain that

$$A(\lambda, I) \leq \frac{1}{2} \mathcal{A}(\lambda - \gamma) w(I^-),$$

that is, if  $\lambda > \gamma$ ,

$$\mathcal{A}(\lambda) \leq \frac{1}{2} \mathcal{A}(\lambda - \gamma).$$

Now, proceeding as in Theorem 3 of [6], it can be obtained (i) of this proposition.

In order to prove (ii), we suppose that  $w \in A_p^-$ ,  $1 < p < \infty$ . Using (3.18), Lemma 3.5 and taking into account that  $w \in D^-$

$$\begin{aligned} & w(\{x \in H_{i,k}^- : |f_{H_{i,k}^+} - f_{I^+}| w(x)^{-1} > \gamma\}) \\ & \leq w\left(\left\{x \in H_{i,k}^- : w(x) < \frac{s}{\gamma} \frac{w(H_{i,k}^+)}{|H_{i,k}^+|}\right\}\right) \\ & \leq C \left[ \frac{s}{\gamma} \frac{w(H_{i,k}^+)}{|H_{i,k}^+|} \frac{|H_{i,k}|}{w(H_{i,k})} \right]^{p'} w(H_{i,k}) \\ & \leq C \left( \frac{s}{\gamma} \right)^{p'} w(H_{i,k}^-). \end{aligned}$$

By (3.15) and (3.14), we have

$$II \leq C \left( \frac{s}{\gamma} \right)^{p'} \sum_{i,k} w(H_{i,k}^-) = C \left( \frac{s}{\gamma} \right)^{p'} \sum_i w(H_i) \leq C \mu \frac{s^{p'-1}}{\gamma^{p'}} w(I^-).$$

Then, (3.16) and (3.17) imply that

$$A(\lambda, I) \leq C \mu \left[ \frac{A(\lambda - \gamma)}{s} + \frac{s^{p'-1}}{\gamma^{p'}} \right] w(I^-).$$

From this inequality, (ii) follows as in Theorem 3 of [6].  $\square$

**Proposition 3.7.** *Let  $0 < \beta < 1$  and  $1 < p < \infty$ . Let  $w$  be a weight such that  $w^{1+\frac{\beta}{1-\beta}p}$  belongs to  $A_p^-$ . Then,  $f \in \mathcal{L}_w(\beta)$  if and only if there exists a constant  $C$  such that (2.1) holds for all bounded interval  $I$  and every  $q : 1 \leq q \leq p'/(1-\beta)$ .*

*Proof:* Suppose that (2.1) holds for every  $q : 1 \leq q \leq p'/(1-\beta)$ . Taking  $q = 1$  it is easy to show that  $f \in \mathcal{L}_w(\beta)$ . Conversely, let  $f$  belong to  $\mathcal{L}_w(\beta)$ . We observe that it will be sufficient to consider  $q = p'/(1-\beta)$ , because from this case and applying Hölder's inequality we obtain (2.1) for every  $1 \leq q < p'/(1-\beta)$ . Given a bounded interval  $I$  and using Proposition 3.3, we have that

$$\begin{aligned} & \int_{I^-} |f(x) - f_{I^+}|^q w(x)^{1-q} dx \\ & \leq \int_{I^-} \left[ \frac{1}{|I^+|} \int_{I^+} |f(x) - f(y)| dy \right]^q w(x)^{1-q} dx \\ & \leq C \int_{I^-} w(x)^{1-q} \left[ \frac{1}{|I^+|} \int_{I^+} \left( \int_x^{x+\frac{|y-x|}{2}} \frac{w(z)}{(z-x)^{1-\beta}} dz \right. \right. \\ & \quad \left. \left. + \int_y^{y+\frac{|y-x|}{2}} \frac{w(z)}{(z-y)^{1-\beta}} dz \right) dy \right]^q dx \\ & \leq C \int_{I^-} w(x)^{1-q} \left( \int_x^{x+\frac{3|I|}{2}} \frac{w(z)}{(z-x)^{1-\beta}} dz \right)^q dx \\ & \quad + \frac{C}{|I^+|^q} \int_{I^-} w(x)^{1-q} \left( \int_{I^+} \int_y^{y+\frac{3|I|}{2}} \frac{w(z)}{(z-y)^{1-\beta}} dz dy \right)^q dx \\ & = A + B. \end{aligned} \tag{3.19}$$



If we denote  $J = I^- \cup I \cup I^+$  then we have the estimate

$$A \leq C \int_{I^-} w(x)^{1-q} I_{\beta}^+(w\chi_J)(x)^q dx.$$

Our hypothesis  $w^{1+\frac{\beta}{1-\beta}p} \in A_p^-$  is equivalent to

$$(3.20) \quad w^{1-\frac{p'}{1-\beta}} \in A_{p'}^+,$$

where  $p' = 1 + \frac{q}{s'}$  and  $\frac{1}{s} = \frac{1}{q} + \beta$ . Then, by Theorem 6 in [4] it follows that

$$A \leq C \left( \int_{-\infty}^{\infty} w(x)^{-\frac{s}{q'}} |w\chi_J(x)|^s dx \right)^{q/s} = C \left( \int_J w(x)^{s/q} dx \right)^{q/s}.$$

Since  $q/s = q\beta + 1 > 1$ , applying Hölder's inequality and taking into account that  $w \in D^-$  we obtain

$$(3.21) \quad A \leq C \int_J w(x) dx |J|^{\frac{q}{s}-1} \leq C w(I^-) |I|^{\beta q}.$$

Let us estimate  $B$ . If we set  $J' = I^+ \cup I^{++} \cup I^{+++}$ , then

$$B \leq \frac{C}{|I^+|^q} \int_{I^-} w(x)^{1-q} \left( \int_{I^+} I_{\beta}^+(w\chi_{J'})(y) dy \right)^q dx.$$

Applying Hölder's inequality,

$$B \leq \frac{C}{|I^+|^q} \left( \int_{I^-} w(x)^{1-q} dx \right) \left( \int_{I^+} w(y) dy \right)^{q/q'} \int_{I^+} w(y)^{1-q} I_{\beta}^+(w\chi_{J'})(y)^q dx.$$

From (3.20), it follows that  $w^{1-q} \in A_q^+$  then, we have that

$$B \leq C \int_{I^+} w(y)^{1-q} I_{\beta}^+(w\chi_{J'})(y)^q dx.$$

Proceeding as in the estimation of  $A$  and taking into account that  $w \in D^-$  we obtain

$$(3.22) \quad B \leq C w(I^-) |I|^{\beta q}.$$

As consequence of (3.19), (3.21) and (3.22) we get (2.1) and the proof of this proposition is complete.  $\square$

*Proof of Theorem 2.2:* We shall prove that  $f$  belonging to  $\mathcal{L}_w(\beta)$  is a sufficient condition for (2.1) holds. The fact that (2.1) is a necessary condition follows as in the previous proposition. For that, we shall consider different cases.

First of all, we assume that  $\beta = 0$  and  $f \in \mathcal{L}_w(0)$ . If  $w \in A_1^-$  we have that (2.1) is an immediate consequence of Proposition 3.6(i). If

$w \in A_p^-$ ,  $1 < p < \infty$ , we have that  $w \in A_{p-\epsilon}^-$  for some  $\epsilon > 0$ . Then, by Proposition 3.6(ii), and proceeding as in Theorem 4 of [6], we obtain that  $f$  satisfies (2.1).

Let  $0 < \beta < 1$  and  $1 < p < \infty$ . Since the weight  $w$  belongs to  $A_p^-$  there exists  $0 < \alpha < \beta$  such that  $w^{1+\frac{\alpha}{1-\alpha}p}$  belongs to  $A_p^-$ . Proceeding as in (3.19), we have that

$$\begin{aligned}
& \int_{I^-} |f(x) - f_{I^+}|^q w(x)^{1-q} dx \\
& \leq C \int_{I^-} w(x)^{1-q} \left[ \frac{1}{|I^+|} \int_{I^+} \left( \int_x^{x+\frac{|y-x|}{2}} \frac{w(z)}{(z-x)^{1-\beta}} dz \right. \right. \\
& \quad \left. \left. + \int_y^{y+\frac{|y-x|}{2}} \frac{w(z)}{(z-y)^{1-\beta}} dz \right) dy \right]^q dx \\
& \leq C |I|^{(\beta-\alpha)q} \int_{I^-} w(x)^{1-q} \left( \int_x^{x+\frac{3|I|}{2}} \frac{w(z)}{(z-x)^{1-\alpha}} dz \right)^q dx \\
& \quad + \frac{C}{|I|^{(\beta-\alpha-1)q}} \int_{I^-} w(x)^{1-q} \left( \int_{I^+} \int_y^{y+\frac{3|I|}{2}} \frac{w(z)}{(z-y)^{1-\alpha}} dz dy \right)^q dx \\
& = |I|^{(\beta-\alpha)q} (A + B).
\end{aligned}$$

Substituting in the proof of the previous proposition  $\alpha$  for  $\beta$  in the estimation of  $A$  and  $B$  we obtain this case.

Finally, we suppose that  $0 < \beta < 1$  and  $p = 1$ . Since the weight  $w$  belongs to  $A_1^-$  it follows that  $w$  belongs to  $A_s^-$  for every  $1 < s < \infty$ . Then, by the previous case we obtain that (2.1) holds for every  $1 \leq q < \infty$ .  $\square$

#### 4. The classes $H^-(\alpha, p)$

The next lemma states necessary conditions for that a weight  $w$  belongs to  $H^-(\alpha, p)$ .

**Lemma 4.1.** *Let  $1 < p \leq \infty$ . If  $w \in H^-(\alpha, p)$  then,*

- (i)  $w^{p'}$  belongs to  $D^-$ ,
- (ii)  $w$  belongs to  $RH^-(p')$ ,
- (iii)  $w$  belongs to  $D^-$ .

*Proof:* The proof of (i) and (ii) are similar to ones of Lemma 3.7 and Lemma 3.8, in [1], respectively. Applying Hölder's inequality and (ii), we obtain (iii).  $\square$

**Lemma 4.2.** *Let  $w$  be a weight. The following conditions are equivalent.*

- (a)  $w \in H^-(\alpha, p)$ .
- (b)  $w \in RH^-(p')$  and there exist positive constants  $C$  and  $\epsilon$  such that,

$$w^{p'}([a, a + \theta t]) \leq C\theta^{(2-\alpha)p' - \epsilon} w^{p'}([a, a + t]),$$

for every  $a \in \mathbb{R}$ ,  $t > 0$  and  $\theta \geq 1$ .

- (c) There exist positive constants  $C$  and  $\epsilon$  such that,

$$\left( \frac{w^{p'}([a, a + \theta t])}{\theta t} \right)^{1/p'} \leq C\theta^{\frac{1}{p} + 1 - \alpha - \frac{\epsilon}{p'}} \frac{w([a - t, a])}{t},$$

for every  $a \in \mathbb{R}$ ,  $t > 0$  and  $\theta \geq 1$ .

*Proof:* (a)  $\Rightarrow$  (b). By Lemma 4.1(ii) we have that  $w \in RH^-(p')$ .

Let  $I = [a, a + t]$ . Applying Hölder's inequality and keeping in mind that  $w \in H^-(\alpha, p)$ ,

$$\begin{aligned} \frac{w^{p'}(I)}{|I|} &\geq \left( \frac{w(I)}{|I|} \right)^{p'} \geq C|I|^{(\frac{1}{p} - \alpha + 1)p'} \int_{a+t}^{\infty} \frac{w(y)^{p'}}{(y-a)^{(2-\alpha)p'}} dy \\ (4.1) \quad &\geq C|I|^{(\frac{1}{p} - \alpha + 1)p'} \sum_{k \geq 0} \frac{1}{(2^{k+1}t)^{(2-\alpha)p'}} \int_{a+2^k t}^{a+2^{k+1}t} w(y)^{p'} dy. \end{aligned}$$

Since  $\sum_{i \geq k} \left( \frac{1}{2^{(2-\alpha)p'}} \right)^i = C \left( \frac{1}{2^{(2-\alpha)p'}} \right)^k$ , by (4.1) and applying Fubini's Theorem,

$$\begin{aligned} \frac{w^{p'}(I)}{|I|} &\geq C|I|^{(\frac{1}{p} - \alpha + 1)p'} \frac{1}{t^{(2-\alpha)p'}} \sum_{k \geq 0} \int_{a+2^k t}^{a+2^{k+1}t} w(y)^{p'} dy \sum_{i \geq k} \left( \frac{1}{2^{(2-\alpha)p'}} \right)^i \\ &= C|I|^{(\frac{1}{p} - \alpha + 1)p'} \sum_{i \geq 0} \frac{1}{(2^i t)^{(2-\alpha)p'}} \sum_{k=0}^i \int_{a+2^k t}^{a+2^{k+1}t} w(y)^{p'} dy \\ &= C|I|^{(\frac{1}{p} - \alpha + 1)p'} \sum_{i \geq 0} \frac{1}{(2^i t)^{(2-\alpha)p'}} \int_{a+t}^{a+2^{i+1}t} w(y)^{p'} dy. \end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{w^{p'}(I)}{|I|} &\geq C|I|^{(\frac{1}{p}-\alpha+1)p'} \sum_{i \geq 0} \frac{1}{(2^i t)^{(2-\alpha)p'}} \int_a^{a+2^{i+1}t} w(y)^{p'} dy \\
&\geq C|I|^{(\frac{1}{p}-\alpha+1)p'} \sum_{i \geq 0} \int_{2^i t}^{2^{i+1}t} \frac{w^{p'}([a, a+s])}{s^{(2-\alpha)p'}} \frac{ds}{s} \\
&= C|I|^{(\frac{1}{p}-\alpha+1)p'} \int_t^\infty \frac{w^{p'}([a, a+s])}{s^{(2-\alpha)p'}} \frac{ds}{s}.
\end{aligned}$$

In consequence,

$$\int_t^\infty \frac{w^{p'}([a, a+s])}{s^{(2-\alpha)p'}} \frac{ds}{s} \leq C \frac{w^{p'}([a, a+t])}{t^{(2-\alpha)p'}}.$$

Now, using Lemma 3.3 in [1] with  $\varphi(s) = w^{p'}([a, a+s])$  and  $r = (2-\alpha)p'$ , there exist  $C$  and  $\epsilon$  such that

$$\varphi(\theta t) \leq C\theta^{r-\epsilon}\varphi(t),$$

for every  $t > 0$  and  $\theta \geq 1$ . That is,

$$w^{p'}([a, a+\theta t]) \leq C\theta^{(2-\alpha)p'-\epsilon}w^{p'}([a, a+t]),$$

for every  $t > 0$  and  $\theta \geq 1$ . This completes the proof of (a)  $\Rightarrow$  (b).

(b)  $\Rightarrow$  (a). Let  $I = [a, a+t]$ . If (b) holds, we have that

$$\begin{aligned}
&\left( \int_{a+t}^\infty \frac{w(y)^{p'}}{(y-a)^{(2-\alpha)p'}} dy \right)^{1/p'} \\
&= \left( \sum_{k=0}^\infty \int_{a+2^k t}^{a+2^{k+1}t} \frac{w(y)^{p'}}{(y-a)^{(2-\alpha)p'}} dy \right)^{1/p'} \\
(4.2) \quad &\leq \left( \sum_{k=0}^\infty \frac{1}{(2^k t)^{(2-\alpha)p'}} w^{p'}([a+t, a+t+2^{k+1}t]) \right)^{1/p'} \\
&\leq C \left( \sum_{k=0}^\infty \frac{(2^{k+1})^{(2-\alpha)p'-\epsilon}}{(2^k t)^{(2-\alpha)p'}} w^{p'}([a+t, a+2t]) \right)^{1/p'} \\
&\leq C \left( \frac{1}{t} \int_{a+t}^{a+2t} w(y)^{p'} dy \right)^{1/p'} t^{\frac{1}{p'}-2+\alpha}.
\end{aligned}$$

Using the hypothesis  $w \in RH^-(p')$  we obtain that (4.2) is bounded by

$$C \frac{1}{t} \int_a^{a+t} w(y) dy t^{\frac{1}{p'}-2+\alpha} = C \frac{w([a, a+t])}{t^{\frac{1}{p}+2-\alpha}},$$

which shows that  $w \in H^-(\alpha, p)$ .

The proof of (b)  $\Rightarrow$  (c) is very simple and we shall omit it.

(c)  $\Rightarrow$  (b). Taking  $\theta = 1$  in (c) we have that  $w \in RH^-(p')$ . Using (c) and Hölder's inequality,

$$\begin{aligned} \left( \frac{w^{p'}([a-t, a+\theta t])}{\theta t} \right)^{1/p'} &= \left( \frac{w^{p'}([a-t, a])}{\theta t} + \frac{w^{p'}([a, a+\theta t])}{\theta t} \right)^{1/p'} \\ &\leq \left( \frac{w^{p'}([a-t, a])}{\theta t} \right)^{1/p'} + C \theta^{\frac{1}{p}+1-\alpha-\frac{\epsilon}{p'}} \left( \frac{w^{p'}([a-t, a])}{t} \right)^{1/p'}. \end{aligned}$$

We can suppose that  $\frac{1}{p} + 1 - \alpha - \frac{\epsilon}{p'} > 0$ , then taking into account that  $\theta \geq 1$

$$\begin{aligned} \left( \frac{w^{p'}([a-t, a-t+\theta t])}{\theta t} \right)^{1/p'} &\leq \left( \frac{w^{p'}([a-t, a+\theta t])}{\theta t} \right)^{1/p'} \\ &\leq C \theta^{\frac{1}{p}+1-\alpha-\frac{\epsilon}{p'}} \left( \frac{w^{p'}([a-t, a])}{t} \right)^{1/p'}. \end{aligned}$$

From these inequalities with  $a = b + t$  we obtain that

$$w^{p'}([b, b+\theta t]) \leq C \theta^{(2-\alpha)p'-\epsilon} w^{p'}([b, b+t]),$$

which completes the proof.  $\square$

*Remark 4.3.* It is easy to see that if  $w^{p'}$  belongs to  $A_1^-$  then,  $w \in H^-(\alpha, p)$ . On the other hand, applying Lemma 4.2 (b)  $\Rightarrow$  (a), it follows that if  $w(x) = |x|^\gamma$  with  $0 < \gamma < 1/p - \alpha + 1$ , then  $w$  belongs to  $H^-(\alpha, p)$ , but  $w$  does not belong to  $A_1^-$ . For  $0 < \alpha < 1/p$ , as an immediate consequence of Lemma 4.2 (c)  $\Rightarrow$  (a) it follows that if  $w^{p'}$  belongs to  $A_{p'+1}^-$  then,  $w$  belongs to  $H^-(\alpha, p)$ .

The next two lemmas show that if  $w$  belongs to  $H^-(\alpha, p)$ ,  $1 < p < \infty$ , then there exists  $\eta > 0$  such that  $w$  belongs to  $H^-(\alpha, q)$  for every  $q : p - \eta < q < p + \eta$ .

**Lemma 4.4.** *Let  $1 < p < \infty$  and  $w \in H^-(\alpha, p)$ . Then, there exists  $\delta_0 \in (0, 1)$  such that  $w \in H^-(\alpha, (p'\delta)')$  for any  $\delta : \delta_0 < \delta \leq 1$ .*

*Proof:* It is a simple variant of Lemma 3.13 in [1].  $\square$

**Lemma 4.5.** *Let  $1 < p < \infty$  and  $w \in H^-(\alpha, p)$ . Then, there exists  $\tau_0 > 1$  such that  $w \in H^-(\alpha, (p'\tau)')$  for any  $1 \leq \tau \leq \tau_0$ .*

*Proof:* Since  $w \in RH^-(p')$  applying Theorem 5.3 in [9], there exists  $\tau_0 > 1$  such that for every  $\tau : 1 \leq \tau \leq \tau_0$  there exists a constant  $C$  such that

$$(4.3) \quad \left( \frac{1}{c-b} \int_b^c w(y)^{p'\tau} dy \right)^{\frac{1}{p'\tau}} \leq C \left( \frac{1}{b-a} \int_a^b w(y) dy \right) \\ \leq C \left( \frac{1}{b-a} \int_a^b w(y)^{p'} dy \right)^{\frac{1}{p'}}$$

for every  $a < b < c$  with  $c - b = 2(b - a)$ . Let  $I = [a, b]$ . Using (4.3) we have that,

$$(4.4) \quad \int_b^\infty \frac{w(y)^{p'\tau}}{(y-a)^{(2-\alpha)p'\tau}} dy \\ = \sum_{k \geq 0} \int_{2^k|I| \leq y-a \leq 2^{k+1}|I|} \frac{w(y)^{p'\tau}}{(y-a)^{(2-\alpha)p'\tau}} dy \\ \leq \sum_{k \geq 0} \frac{1}{(2^k|I|)^{(2-\alpha)p'\tau}} \int_{2^k|I| \leq y-a \leq 2^{k+1}|I|} w(y)^{p'\tau} dy \\ \leq C \sum_{k \geq 0} \frac{1}{(2^k|I|)^{(2-\alpha)p'\tau-1}} \left( \frac{1}{2^k|I|} \int_{2^{k-1}|I| \leq y-a \leq 2^k|I|} w(y)^{p'} dy \right)^\tau.$$

Taking into account that  $\tau > 1$ , (4.4) is bounded by

$$C \sum_{k \geq 0} 2^k |I| \left( \frac{1}{2^k |I|} \int_{2^{k-1}|I| \leq y-a \leq 2^k|I|} \frac{w(y)^{p'}}{(y-a)^{(2-\alpha)p'}} dy \right)^\tau \\ \leq C |I|^{1-\tau} \left( \int_{\frac{|I|}{2} \leq y-a} \frac{w(y)^{p'}}{(y-a)^{(2-\alpha)p'}} dy \right)^\tau.$$

Keeping in mind that  $w \in H^-(\alpha, p)$  we have,

$$\begin{aligned} \int_b^\infty \frac{w(y)^{p'\tau}}{(y-a)^{(2-\alpha)p'\tau}} dy &\leq C|I|^{1-\tau} \left( \frac{w([a, a+|I|/2])}{|I|} |I|^{-1/p+\alpha-1} \right)^{p'\tau} \\ &= C \left( \frac{w(I)}{|I|} \frac{1}{|I|^{\frac{1}{(p'\tau)'}-\alpha+1}} \right)^{p'\tau}, \end{aligned}$$

which implies that  $w \in H^-(\alpha, (p'\tau)')$ .  $\square$

**Lemma 4.6.** *Let  $1 < p_1 < p_2 < \infty$ . Suppose that  $w \in H^-(\alpha, p_i)$  for  $i = 1, 2$ . Then  $w \in H^-(\alpha, p)$  for every  $p : p_1 < p < p_2$ .*

*Proof:* This is an one-sided version of Lemma 3.15 in [1].  $\square$

**Lemma 4.7.** *Let  $1 < p < \infty$  and  $w \in RH^-(p')$ . There exists a constant  $C$  such that for every  $f \in \widetilde{L}_w^p$  and every bounded interval  $I = [a, b]$ , if we denote  $\tilde{I}^- = [a - \frac{|I|}{2}, a]$  then,*

$$\int_I |f(x)| dx \leq C \frac{w(\tilde{I}^-)}{|I|^{1/p}} [f]_{p,w}.$$

*Proof:* Since  $w \in RH^-(p')$  by Theorem 5.3 in [9], there exists  $s > p'$  such that  $w \in RH^-(s)$ , that is, there exists a constant  $C$  such that for every bounded interval  $I$ ,

$$\left( \frac{1}{|I|} \int_I w(x)^s dx \right)^{1/s} \leq C \frac{w(\tilde{I}^-)}{|I|}.$$

From this fact, the proof follows as in Lemma 4.1 of [1].  $\square$

**Lemma 4.8.** *Let  $1 < p < \infty$  and  $w \in H^-(\alpha, p)$ . Then there exists a constant  $C$  such that for every  $f \in \widetilde{L}_w^p$  and every bounded interval  $I = [a, b]$ ,*

$$\int_b^\infty \frac{|f(y)|}{(y-a)^{2-\alpha}} dy \leq C \frac{w(I)}{|I|^{2+\frac{1}{p}-\alpha}} [f]_{p,w}.$$

*Proof:* Taking into account Lemma 4.4 and Lemma 4.5, the proof of this lemma is similar to one in Lemma 4.4 of [1].  $\square$

**Lemma 4.9.** *Let  $\alpha > 0$  and  $\delta \geq 0$  such that  $0 < \alpha + \delta < 1$ . Let  $w \in D^-$ . For  $a < b$ , we denote  $c = \frac{a+b}{2}$  and  $I = [c, b]$ . Then, for every  $f \in \mathcal{L}_w(\delta)$ , there exists a constant  $C$  such that,*

$$\begin{aligned} \text{(i)} \quad & \int_b^\infty \frac{|f(y) - f_I|}{(y-a)^{2-\alpha}} dy \leq C \|f\|_{\mathcal{L}_w(\delta)} \int_c^\infty \frac{w(y)}{(y-a)^{2-\alpha-\delta}} dy. \\ \text{(ii)} \quad & \int_a^b \frac{|f(y) - f_I|}{(y-a)^{1-\alpha}} dy \leq C \|f\|_{\mathcal{L}_w(\delta)} \int_a^c \frac{w(y)}{(y-a)^{1-\alpha-\delta}} dy. \end{aligned}$$

*Proof:* The proof of (i) and (ii) are similar, then we only prove (i).

For every  $j \geq 0$ , let  $I_j = [a + 2^j|I|, a + 2^{j+1}|I|]$ . We observe that  $I_0 = [a + |I|, a + 2|I|] = [c, b] = I$ . Since  $f \in \mathcal{L}_w(\delta)$  we have that,

$$\begin{aligned} \int_b^\infty \frac{|f(y) - f_I|}{(y-a)^{2-\alpha}} dy &= \sum_{j=1}^\infty \int_{a+2^j|I|}^{a+2^{j+1}|I|} \frac{|f(y) - f_I|}{(y-a)^{2-\alpha}} dy \\ &\leq \sum_{j=1}^\infty \frac{1}{(2^j|I|)^{2-\alpha}} \int_{a+2^j|I|}^{a+2^{j+1}|I|} |f(y) - f_{I_0}| dy \\ &\leq \sum_{j=1}^\infty \frac{1}{(2^j|I|)^{2-\alpha}} \left[ \int_{a+2^j|I|}^{a+2^{j+1}|I|} |f(y) - f_{I_j}| dy \right. \\ &\quad \left. + 2^j|I| \sum_{k=1}^j |f_{I_k} - f_{I_{k-1}}| \right] \\ &\leq \sum_{j=1}^\infty \frac{1}{(2^j|I|)^{1-\alpha}} \left[ C \|f\|_{\mathcal{L}_w(\delta)} w(I_j) (2^j|I|)^{\delta-1} \right. \\ &\quad \left. + \sum_{k=1}^j \frac{1}{|I_{k-1}|} \int_{I_{k-1}} |f(y) - f_{I_k}| dy \right]. \end{aligned} \tag{4.5}$$

Using that  $f \in \mathcal{L}_w(\delta)$  and  $w \in D^-$  we obtain the estimate,

$$\frac{1}{|I_{k-1}|} \int_{I_{k-1}} |f(y) - f_{I_k}| dy \leq C \|f\|_{\mathcal{L}_w(\delta)} w(I_{k-1}) (2^{k-1}|I|)^{\delta-1}.$$



Then applying Fubini's Theorem, (4.5) is bounded by

$$\begin{aligned}
& C\|f\|_{\mathcal{L}_w(\delta)} \sum_{j=1}^{\infty} \frac{1}{(2^j|I|)^{1-\alpha}} \sum_{k=0}^j w(I_k)(2^k|I|)^{\delta-1} \\
&= C\|f\|_{\mathcal{L}_w(\delta)} \sum_{k=0}^{\infty} w(I_k)(2^k|I|)^{\delta-1} \sum_{j=k}^{\infty} \frac{1}{(2^j|I|)^{1-\alpha}} \\
&= C\|f\|_{\mathcal{L}_w(\delta)} \sum_{k=0}^{\infty} \frac{1}{(2^k|I|)^{2-\alpha-\delta}} \int_{a+2^k|I|}^{a+2^{k+1}|I|} w(y) dy \\
&\leq C\|f\|_{\mathcal{L}_w(\delta)} \int_c^{\infty} \frac{w(y)}{(y-a)^{2-\alpha-\delta}} dy,
\end{aligned}$$

as we wanted to prove.  $\square$

## 5. Proof of Theorems 2.3 and 2.4

*Proof of Theorem 2.3:* (i)  $\Rightarrow$  (ii). Let  $w \in H^-(\alpha, p)$  and  $x_0 \in \mathbb{R}$ . Given  $f \in \widetilde{L}_w^p$  let  $\widetilde{I}_\alpha^+(f)$  define as in (2.2). Choose a bounded interval  $I = [a, a+h]$ . We consider  $I_0 = [a+2h, x_0]$  if  $a+2h \leq x_0$  and  $I_0 = \emptyset$  if  $x_0 < a+2h$ , and we also define  $I_1 = [x_0, a+2h]$  if  $x_0 < a+2h$  and  $I_1 = \emptyset$  in the other case. We set

$$a_I = \int_{I_0} \frac{f(y)}{(y-a)^{1-\alpha}} dy + \int_{x_0}^{\infty} \left[ \frac{1 - \chi_{I_1}(y)}{(y-a)^{1-\alpha}} - \frac{1 - \chi_{[x_0, x_0+1]}(y)}{(y-x_0)^{1-\alpha}} \right] f(y) dy.$$

We shall show that  $a_I$  is a finite constant.

Suppose that  $x_0 < a+2h$ . Let  $n$  be a positive integer such that  $a+2^n h > x_0+1$  and  $|a-x_0| \leq 2^{n-1}h$ . Then,

$$\begin{aligned}
a_I &= \left( \int_{x_0}^{a+2^n h} + \int_{a+2^n h}^{\infty} \right) \left[ \frac{1 - \chi_{[x_0, a+2h]}(y)}{(y-a)^{1-\alpha}} - \frac{1 - \chi_{[x_0, x_0+1]}(y)}{(y-x_0)^{1-\alpha}} \right] f(y) dy \\
&= J_1 + J_2.
\end{aligned}$$

For each  $y \geq a+2^n h$ , by Mean Value Theorem, there exists  $\theta : 0 < \theta < 1$  such that,

$$\left| \frac{1}{(y-a)^{1-\alpha}} - \frac{1}{(y-x_0)^{1-\alpha}} \right| \leq C \frac{|x_0-a|}{|y-\theta a - (1-\theta)x_0|^{2-\alpha}} \leq C \frac{|x_0-a|}{|y-a|^{2-\alpha}}.$$

Then, applying Lemma 4.8, we have that

$$|J_2| \leq C|x_0 - a| \int_{a+2^n h}^{\infty} \frac{|f(y)|}{|y - a|^{2-\alpha}} dy \leq C|x_0 - a| \frac{w([a, a + 2^n h])}{(2^n h)^{2+\frac{1}{p}-\alpha}} [f]_{p,w} < \infty.$$

On the other hand, since  $f \in \widetilde{L}_w^p$  and using Lemma 4.7, we get

$$\begin{aligned} |J_1| &\leq \int_{a+2h}^{a+2^n h} \frac{|f(y)|}{(y-a)^{1-\alpha}} dy + \int_{x_0+1}^{a+2^n h} \frac{|f(y)|}{(y-x_0)^{1-\alpha}} dy \\ &\leq \frac{1}{(2h)^{1-\alpha}} \int_{a+2h}^{a+2^n h} |f(y)| dy + \int_{x_0+1}^{a+2^n h} |f(y)| dy < \infty. \end{aligned}$$

The case  $x_0 \geq a + 2h$  can be proved in a similar way.

Now, let

$$\begin{aligned} (5.1) \quad A(x) &= \int_x^{a+2h} \frac{f(y)}{(y-x)^{1-\alpha}} dy \\ &\quad + \int_{a+2h}^{\infty} \left[ \frac{1}{(y-x)^{1-\alpha}} - \frac{1}{(y-a)^{1-\alpha}} \right] f(y) dy \\ &= A_1(x) + A_2(x). \end{aligned}$$

It follows that,

$$(5.2) \quad \widetilde{I}_\alpha^+(f)(x) = A(x) + a_I.$$

We shall show that,

$$\int_I |\widetilde{I}_\alpha^+(f)(x) - a_I| dx \leq C|I|^{\alpha-1/p} w(I^-) [f]_{p,w}.$$

We observe that taking into account (5.2) and (5.1) it is sufficient to prove that

$$\int_I |A_j(x)| dx \leq C|I|^{\alpha-1/p} w(I^-) [f]_{p,w},$$

for  $j = 1, 2$ . Applying Mean Value Theorem, Lemma 4.8 and Lemma 4.1(iii) for every  $x \in I = [a, a + h]$  we have that,

$$\begin{aligned} |A_2(x)| &\leq \int_{a+2h}^{\infty} \left| \frac{1}{(y-x)^{1-\alpha}} - \frac{1}{(y-a)^{1-\alpha}} \right| |f(y)| dy \\ &\leq Ch \int_{a+2h}^{\infty} \frac{|f(y)|}{|y-a|^{2-\alpha}} dy \leq Ch \frac{w([a, a + 2h])}{(2h)^{2+\frac{1}{p}-\alpha}} [f]_{p,w} \\ &\leq C \frac{w([a-h, a])}{h^{1+\frac{1}{p}-\alpha}} [f]_{p,w}. \end{aligned}$$

Therefore,

$$\int_I |A_2(x)| dx \leq C |I|^{\alpha-1/p} w(I^-)[f]_{p,w}.$$

With respect to  $A_1(x)$ , changing the order of integration and applying Lemma 4.7,

$$\begin{aligned} \int_a^{a+h} |A_1(x)| dx &\leq \int_a^{a+h} \int_x^{a+2h} \frac{|f(y)|}{(y-x)^{1-\alpha}} dy dx \\ &\leq \int_a^{a+2h} |f(y)| \int_a^y \frac{dx}{(y-x)^{1-\alpha}} dy \\ &\leq Ch^\alpha \int_a^{a+2h} |f(y)| dy \\ &\leq Ch^{\alpha-1/p} w([a-h, a])[f]_{p,w}, \end{aligned}$$

which completes the proof of (i)  $\Rightarrow$  (ii).

The implication (ii)  $\Rightarrow$  (iii) is obvious.

(iii)  $\Rightarrow$  (i). Let  $a \in \mathbb{R}$  and  $h > 0$ . We consider  $f \geq 0$  such that  $\text{sop}(f) \subseteq [a+4h, \infty)$ . For each  $x \in [a, a+h]$  we have that,

$$\begin{aligned} |I_\alpha^+(f)(x) - I_\alpha^+(f)_{[a+2h, a+3h]}| \\ = \frac{1}{h} \int_{a+2h}^{a+3h} \int_{a+4h}^\infty f(y) \left[ \frac{1}{(y-t)^{1-\alpha}} - \frac{1}{(y-x)^{1-\alpha}} \right] dy dt. \end{aligned}$$

Applying Mean Value Theorem, for each  $y \geq a+4h$  we obtain,

$$\frac{1}{(y-t)^{1-\alpha}} - \frac{1}{(y-x)^{1-\alpha}} \geq C \frac{|x-t|}{(y-a)^{2-\alpha}} \geq C \frac{h}{(y-a)^{2-\alpha}}.$$

In consequence,

$$|I_\alpha^+(f)(x) - I_\alpha^+(f)_{[a+2h, a+3h]}| \geq Ch \int_{a+4h}^\infty \frac{f(y)}{(y-a)^{2-\alpha}} dy.$$

Then, if  $f \in L_w^p$ , using (iii) we have that,

$$\begin{aligned} Ch^2 \int_{a+4h}^\infty \frac{f(y)}{(y-a)^{2-\alpha}} dy &\leq 2 \int_a^{a+3h} |I_\alpha^+(f)(x) - I_\alpha^+(f)_{[a, a+3h]}| dx \\ &\leq C(3h)^\beta w([a-3h, a]) \left[ \int \left( \frac{f(y)}{w(y)} \right)^p dy \right]^{1/p}. \end{aligned}$$

Now, taking into account that  $\beta = \alpha - 1/p$  it follows that,

$$(5.3) \quad h^{1/p-\alpha+1} \int_{a+4h}^{\infty} \frac{f(y)}{(y-a+3h)^{2-\alpha}} dy \\ \leq C \frac{w([a-3h, a+4h])}{h} \left[ \int_{a+4h}^{\infty} \left( \frac{f(y)}{w(y)} \right)^p dy \right]^{1/p}.$$

For each  $m > 2$  we put,

$$f_m(y) = \frac{w(y)^{p'}}{(y-a+3h)^{\frac{2-\alpha}{p-1}}} \chi_{[a+4h, a+2^m h]}(y) \chi_{\{0 \leq w \leq m\}}(y).$$

It is easy to check that  $f_m \in L_w^p$ . Using (5.3) with  $f_m$  and taking the limit, we obtain that

$$h^{1/p-\alpha+1} \left( \int_{a+4h}^{\infty} \frac{w(y)^{p'}}{(y-a+3h)^{(2-\alpha)p'}} dy \right)^{1/p'} \leq C \frac{w([a-3h, a+4h])}{h},$$

which shows that  $w \in H^-(\alpha, p)$ .  $\square$

*Remark 5.1.* By Theorem 2.1, if  $0 \leq \beta < 1$ , we can substitute in Theorem 2.3,  $\mathcal{L}_w(\beta)$  for  $\mathcal{L}_w^-(\beta)$ . That is not possible for  $-1 < \beta < 0$ . In fact, if  $w$  and  $f$  are defined as in Remark 3.2(ii), then

$$I_{\alpha}^{+}(f)(x) = \begin{cases} \frac{\Gamma(\alpha)}{a^{\alpha}} e^{-ax}, & x \geq 0 \\ \frac{|x|^{\alpha}}{\alpha} + \frac{e^{-ax}}{a^{\alpha}} \int_{a|x|}^{\infty} e^{-u} u^{\alpha-1} du, & x < 0. \end{cases}$$

Therefore, the same arguments used in Remark 3.2 imply that  $I_{\alpha}^{+}(f)$  does not belong to  $\mathcal{L}_w(\beta)$ .

*Proof of Theorem 2.4:* (i)  $\Rightarrow$  (ii). Let  $N$  be a positive integer. For any integer  $a$  applying Fubini's Theorem and taking into account that  $w$  is a locally integrable function, we have that

$$\int_a^{a+1} \int_x^{x+N} \frac{w(y)}{(y-x)^{1-\alpha}} dy dx < \infty.$$

In consequence, for almost every  $x$  and every positive integer  $N$

$$(5.4) \quad \int_x^{x+N} \frac{w(y)}{(y-x)^{1-\alpha}} dy < \infty.$$

Let  $x_0$  satisfying (5.4). We consider

$$(5.5) \quad \widetilde{I}_{\alpha}^{+}(f)(x) = \int_{-\infty}^{\infty} \left[ \frac{\chi_{[x_0, \infty)}(y)}{|y-x_0|^{1-\alpha}} - \frac{\chi_{[x, \infty)}(y)}{|y-x|^{1-\alpha}} \right] f(y) dy.$$

We shall show that if  $f \in \mathcal{L}_w(0)$  then  $\widetilde{I}_\alpha^+(f)$ , defined as in (5.5), is finite for every  $x$  satisfying (5.4). Fix  $x$  satisfying (5.4). Suppose that  $x_0 < x$  and let  $R \in \mathbb{Q} : x_0 < x \leq x_0 + R/4$ . We consider the interval  $I = [x_0 + R/2, x_0 + R]$ . Taking into account that the function  $g(y) = \frac{\chi_{[x_0, \infty)}(y)}{|y - x_0|^{1-\alpha}} - \frac{\chi_{[x, \infty)}(y)}{|y - x|^{1-\alpha}}$  is integrable and  $\int_{-\infty}^{\infty} g(y) dy = 0$  we can write,

$$\begin{aligned} \widetilde{I}_\alpha^+(f)(x) &= \int_{-\infty}^{\infty} \left[ \frac{\chi_{[x_0, \infty)}(y)}{|y - x_0|^{1-\alpha}} - \frac{\chi_{[x, \infty)}(y)}{|y - x|^{1-\alpha}} \right] [f(y) - f_I] dy \\ &= I_1(x) + I_2(x), \end{aligned}$$

where,

$$I_1(x) = \int_{x_0}^{x_0+R} \quad \text{and} \quad I_2(x) = \int_{x_0+R}^{\infty}.$$

We shall prove that

$$\begin{aligned} (5.6) \quad & |\widetilde{I}_\alpha^+(f)(x)| \\ & \leq C \|f\|_{\mathcal{L}_w(0)} \left[ \int_{x_0}^{x_0+5R/4} \frac{w(y)}{(y - x_0)^{1-\alpha}} dy + \int_x^{x+5R/4} \frac{w(y)}{(y - x)^{1-\alpha}} dy \right]. \end{aligned}$$

We observe that,

$$|I_1(x)| \leq \int_{x_0}^{x_0+R} \frac{|f(y) - f_I|}{|y - x_0|^{1-\alpha}} dy + \int_x^{x_0+R} \frac{|f(y) - f_I|}{|y - x|^{1-\alpha}} dy.$$

Let  $J = [x + R/2, x + R]$ . Applying Lemma 4.9(ii) we have that

$$\begin{aligned} (5.7) \quad |I_1(x)| &\leq \int_{x_0}^{x_0+R} \frac{|f(y) - f_I|}{|y - x_0|^{1-\alpha}} dy \\ &\quad + \int_x^{x+R} \frac{|f(y) - f_J|}{|y - x|^{1-\alpha}} + |f_I - f_J| \int_x^{x+R} \frac{dy}{|y - x|^{1-\alpha}} \\ &\leq C \|f\|_{\mathcal{L}_w(0)} \int_{x_0}^{x_0+R/2} \frac{w(y)}{(y - x_0)^{1-\alpha}} dy \\ &\quad + C \|f\|_{\mathcal{L}_w(0)} \int_x^{x+R/2} \frac{w(y)}{(y - x)^{1-\alpha}} dy \\ &\quad + \frac{R^\alpha}{\alpha} |f_I - f_J|. \end{aligned}$$

Since  $x_0 < x < x_0 + R/4$  and  $f \in \mathcal{L}_w(0)$  we have,

$$R^\alpha |f_I - f_J| \leq C \|f\|_{\mathcal{L}_w(0)} \int_{x_0}^{x_0+5/4R} \frac{w(y)}{(y-x_0)^{1-\alpha}} dy.$$

Then, by (5.7)

$$|I_1(x)| \leq C \|f\|_{\mathcal{L}_w(0)} \left[ \int_{x_0}^{x_0+5R/4} \frac{w(y)}{(y-x_0)^{1-\alpha}} dy + \int_x^{x+5R/4} \frac{w(y)}{(y-x)^{1-\alpha}} dy \right].$$

Now, let us estimate  $I_2$ . Applying Mean Value Theorem,

$$\begin{aligned} |I_2(x)| &\leq \int_{x_0+R}^{\infty} \left| \frac{1}{|y-x_0|^{1-\alpha}} - \frac{1}{|y-x|^{1-\alpha}} \right| |f(y) - f_I| dy \\ &\leq C |x_0 - x| \int_{x_0+R}^{\infty} \frac{|f(y) - f_{[x_0+R/2, x_0+R]}|}{(y-x_0)^{2-\alpha}} dy. \end{aligned}$$

Using Lemma 4.9(i) and taking into account that  $w \in H^-(\alpha, \infty)$  we get,

$$\begin{aligned} |I_2(x)| &\leq CR \|f\|_{\mathcal{L}_w(0)} \int_{x_0+R/2}^{\infty} \frac{w(y)}{(y-x_0)^{2-\alpha}} dy \\ &\leq CR \|f\|_{\mathcal{L}_w(0)} \frac{w([x_0, x_0 + R/2])}{R^{2-\alpha}} \\ &\leq C \|f\|_{\mathcal{L}_w(0)} \int_{x_0}^{x_0+R/2} \frac{w(y)}{(y-x_0)^{1-\alpha}} dy. \end{aligned}$$

Then, if  $x_0 < x < x_0 + R/4$  or in the case  $x_0 - R/4 < x < x_0$ , we have that (5.6) holds. Since  $\mathbb{R} = \cup_{R \in \mathbb{Q}_{>0}} [x_0 - R/4, x_0 + R/4]$ , it follows that  $\widetilde{I_\alpha^+}(f)(x)$  is finite for almost every  $x$ .

Let us show that  $\widetilde{I_\alpha^+}(f) \in \mathcal{L}_w(\alpha)$ . For almost every  $x_1 < x_2$ , if we define  $R = 4|x_1 - x_2|$ , we have that  $x_1 < x_2 \leq x_1 + R/4$  and using (5.6)

we get

$$\begin{aligned}
& |\widetilde{I_\alpha^+}(f)(x_1) - \widetilde{I_\alpha^+}(f)(x_2)| \\
& \leq \int_{-\infty}^{\infty} \left| \frac{\chi_{[x_1, \infty)}(y)}{(y - x_1)^{1-\alpha}} - \frac{\chi_{[x_2, \infty)}(y)}{(y - x_2)^{1-\alpha}} \right| |f(y) - f_{[x_1+R/2, x_1+R]}| dy \\
& \leq C \|f\|_{\mathcal{L}_w(0)} \left[ \int_{x_1}^{x_1+5|x_1-x_2|} \frac{w(y)}{(y - x_1)^{1-\alpha}} dy \right. \\
& \quad \left. + \int_{x_2}^{x_2+5|x_1-x_2|} \frac{w(y)}{(y - x_2)^{1-\alpha}} dy \right].
\end{aligned}$$

Taking into account that  $w \in D^-$  and using Proposition 3.3 it follows that  $\widetilde{I_\alpha^+}(f) \in \mathcal{L}_w(\alpha)$ .

(ii)  $\Rightarrow$  (i). This implication is similar to (iii)  $\Rightarrow$  (i) in Theorem 2.3.  $\square$

**Corollary 5.2.** *Let  $\alpha, \delta \in \mathbb{R}^+$  such that  $0 < \alpha + \delta < 1$ . The following statements are equivalent.*

- (a)  $w \in H^-(\delta, \infty)$  and the operator  $I_\alpha$  can be extended to a linear bounded operator  $\widetilde{I_\alpha^+} : \mathcal{L}_w(\delta) \rightarrow \mathcal{L}_w(\alpha + \delta)$ .
- (b)  $w \in H^-(\alpha + \delta, \infty)$ .

*Proof:* The proof is a simple variant of Corollary 2.12 in [1].  $\square$

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